Finite-size scaling as a cross-over phenomena

One extremely useful application of the cross-over phenomena is to finite-sized systems, e.g., systems studied in computer simulations. So far our discussion has been restricted to thermodynamic limit, $L=\infty$. In a finite system, there can't be any true spontaneous symmetry breaking or associated critical phenomena. Curiously, one can think of $\frac{1}{L}$ as a relevant perturbation which cuts off the singularities associated with $L=\infty$ critical phenomena.

Under coarse graining by a factor b, the effective system size is decreased by the factor b

$$L_{eff} \to \frac{L_{eff}}{b} \Rightarrow L^{-1}_{eff} \to b\, L^{-1}_{eff} \Rightarrow$$

the RG eigenvalue associated with $\frac{1}{L}$ is unity. Another way to deduce this is to note that free energy density will depend on the total system size L^d via $L_{eff} = \frac{L}{a}$ where a is the lattice cutoff. Under RG transformation $a \to b a$ while L is fixed. $\Rightarrow L_{eff}^{-1} \to L_{eff}^{-1} b$.

From now on, lets set $a = 1$, so that $L_{eff} \equiv L$.

Consider again an Ising system,

$$f_s = b^{-d} f_s(t b^{y_t}, h b^{y_h}, L^{-1} b)$$

For simplicity, consider the case $h=0$ and lets study the specific heat $\Rightarrow$

$$c \sim \frac{\partial^2 f_s}{\partial t^2} \sim b^{-d+2y_t} \phi(t b^{y_t}, L^{-1} b)$$

Setting $t b^{y_t} \sim 1 \Rightarrow c \sim t^{\frac{d}{y_t} - 2} \phi(L^{-1} t^{-1/y_t})$

When $L^{-1} t^{-1/y_t} \ll 1$, i.e. $L \gg t^{-1/y_t} = \xi_\infty$,

where the subscript ∞ on ξ_∞ denotes that it is the correlation length of a would-be infinite system, one will observe the true critical phenomena associated with $L = \infty$ system, i.e.

$$c \sim t^{d/y_t - 2} = t^{-\alpha}.$$

However, when $L \lesssim \xi_\infty$, the true correlation length will be cut-off by the system size, $\xi \sim L$, and therefore, the specific heat cannot diverge as $t \to 0$. Therefore, we require that $\phi(x)$ as $x \to \infty$ be such that c has a finite value. Writing $\phi(x) \sim x^a$ as $x \to \infty$

$$\Rightarrow \quad \frac{-a}{y_t} + \frac{d}{y_t} - 2 = 0$$

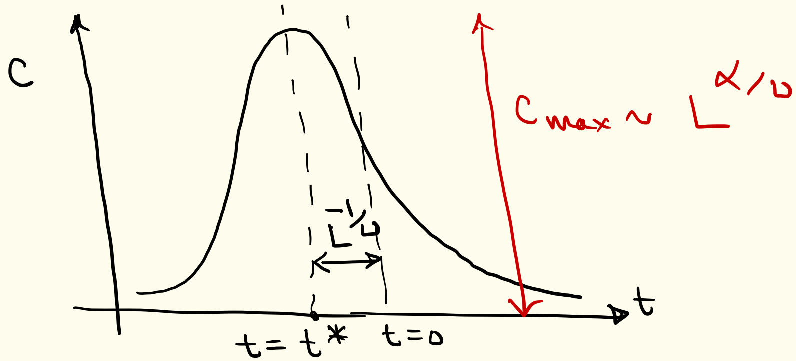
$$\Rightarrow \quad a = d - 2y_t$$

$$\Rightarrow \quad c \sim L^{(2y_t - d)} \quad \text{in this limit.}$$

$$\sim L^{\alpha/\nu}$$

Therefore, in a finite-size system, the specific heat has a peak of the size $L^{\alpha/\nu}$. The location of the peak as a f^n of t is determined by the maximum of the f^n $\phi(x)$. Assuming $\phi(x)$ has a maximum at some value $x = x_0$ $\Rightarrow$ the peak will occur at $L^{-1} t^{* -1/y_t} = x_0$

$\Rightarrow$ $t^* \sim L^{-y_t} \sim L^{-1/\nu}$ i.e. not at $t=0$ but a little-bit shifted.



The direction of the shift is non-universal. Periodic boundary conditions depress fluctuations since the wave-vectors $\sim \frac{2\pi n}{L}$ are restricted.

Therefore, $t^*_{PBC} > 0$ while open boundary conditions allow for more fluctuations and $t^*_{OBC} < 0$.

Finite size scaling:

The above discussion leads to useful ways to perform data collapse in a finite-sized system. We discuss two different schemes:

(a) Finite-size scaling of correlation length:

The correlation length must scale as

$$\xi = b\,\xi(t\,b^{y_t},\ L^{-1} b)$$

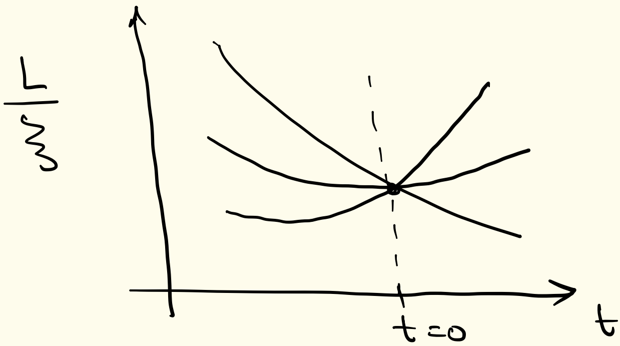
$$= t^{-1/y_t}\,\xi(L^{-1} t^{-1/y_t})$$

$$= L\ \phi(L^{y_t} t)$$

Again, when $L t^{1/y_t} \gg 1$ $\phi(x) \to \frac{1}{x^{1/y_t}}$, so that $\xi \sim t^{-1/y_t}$. On the other hand, when $t L^{y_t} \ll 1$ $\phi(x) \to$ const so that $\xi \sim L$. Thus, in this limit $\phi(x)$ does not diverge and is expected to be analytic. Taylor expanding, $\xi = L\,[c + t L^{y_t} + \cdots]$

$$\Rightarrow \frac{L}{\xi} \sim \text{const.} + t\,L^{y_t} + O(t^2)$$

This expression implies that if one plots $\frac{L}{\xi}$ as a f^n of t for different values of L, all plots take an L independent value at $t=0$! Therefore one can figure out T_c using this scheme.



One can also obtain the exponent $y_t = \frac{1}{\nu}$ using $\log\left[\frac{\partial}{\partial t}\left(\frac{L}{\xi}\right)\right]\Big|_{t=0} \sim \frac{1}{\nu}\log L$.

(b) finite-size scaling of magnetization:

Following same strategy as above, magnetization m in a finite size system takes the form

$$m \sim L^{-\beta/\nu} \, \phi(t L^{1/\nu})$$

when $x \gg 1$ $\phi(x) \sim x^{\beta}$ while when $|x| \ll 1$, $\phi(x) \to$ const, so that m vanishes in a finite system. One can also consider various moments of magnetization, e.g.

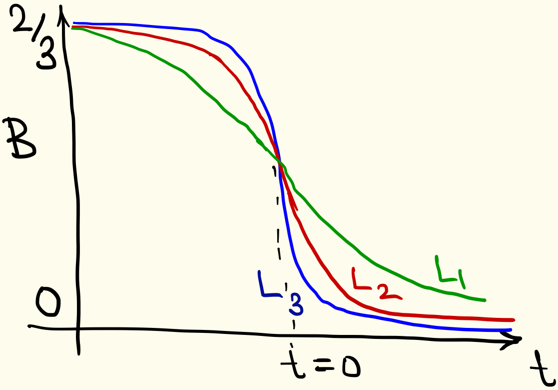
$$\langle m^n \rangle \sim L^{-4\beta/\nu} \, \phi_n(t L^{1/\nu}).$$

Consider the following quantity ('Binder cumulant')

$$B = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2}$$

$$= 1 - \frac{\phi_4(t L^{1/\nu})}{3 \left[\phi_2(t L^{1/\nu})\right]^2}$$

This quantity is a f^n only of the scaling variable $t L^{1/D}$, and not t and L separately. Therefore, plotting B as a f^n of t for different L all cross at $t = 0$



Further, for $T > T_c$ $\langle m^4 \rangle = 3 \langle m^2 \rangle^2$ $\Rightarrow$ $B = 0$ while for $T < T_c$ $\langle m^4 \rangle = \langle m^2 \rangle^2$ $\Rightarrow$ $B = 2/3$.